Madras College Maths Department
Higher Maths
Apps 1.1 Straight Line

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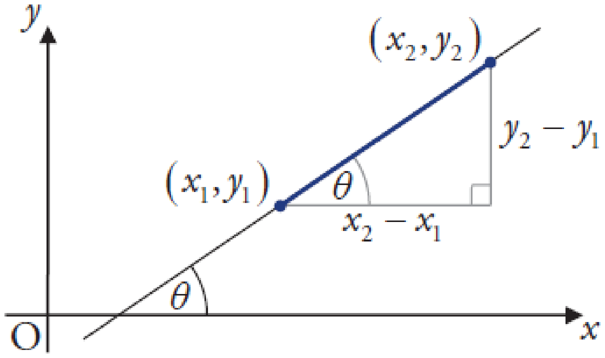
Written solutions for each exercise are available at

http://madrasmaths.com/courses/higher/revison_materials_higher.html

You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

Gradients A

Consider a straight line passing through the points (x_1, y_1) and (x_2, y_2):



Note
"θ" is the Greek letter "theta".
It is often used to stand for an angle.

The **gradient** m of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{change in vertical height}}{\text{change in horizontal distance}} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{for } x_1 \neq x_2.$$

Note

As a result of the above definitions:

- lines with positive gradients slope up, from left to right;
- lines with negative gradients slope down, from left to right;
- lines parallel to the x-axis have a gradient of zero;
- lines parallel to the y-axis have an undefined gradient.

We may also use the fact that:

Two distinct lines are said to be **parallel** when they have the same gradient (or when both lines are vertical).

Parallel Lines

Parallel lines have equal gradients.

1 Consider the points S(2, 5), T(5,9), U-3, -4) and V(3, 4). Show that ST and UV are parallel.

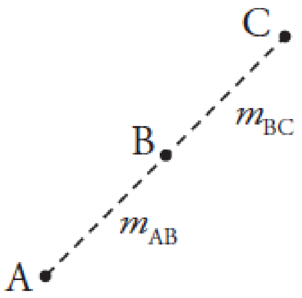
2 Find the equation of the line GH which passes through the point (4, -2) and is parallel to the line RS with equation $3x + 2y = 7$.

Collinearity

Points which lie on the same straight line are said to be **collinear**.

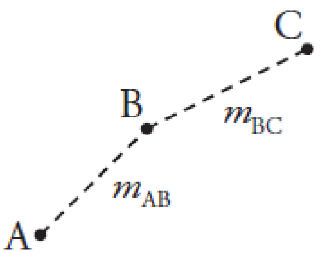
To test if three points A, B and C are collinear we can:

1. Work out m_{AB}.
2. Work out m_{BC} (or m_{AC}).
3. If the gradients from 1. and 2. are the same then A, B and C are collinear.



$m_{AB} = m_{BC}$ so A, B and C are collinear.

If the gradients are different then the points are not collinear.



$m_{AB} \neq m_{BC}$ so A, B and C are not collinear.

This test for collinearity can only be used in two dimensions.

EXAMPLES

1. Show that the points $P(-6, -1)$, $Q(0, 2)$ and $R(8, 6)$ are collinear.

2. The points $A(1,-1)$, $B(-1,k)$ and $C(5,7)$ are collinear.
   Find the value of k.

Gradients of Perpendicular Lines

Two lines at right-angles to each other are said to be **perpendicular**.

If perpendicular lines have gradients m and $m_{\perp}$ then

$$m \times m_{\perp} = -1 .$$

Conversely, if $m \times m_{\perp} = -1$ then the lines are perpendicular.

The simple rule is: if you know the gradient of one of the lines, then the gradient of the other is calculated by inverting the gradient (i.e. "flipping" the fraction) and changing the sign. For example:

$$\text{if} \quad m = \tfrac{2}{3} \quad \text{then} \quad m_{\perp} = -\tfrac{3}{2}.$$

Note that this rule *cannot* be used if the line is parallel to the x- or y-axis.

- If a line is parallel to the x-axis $(m = 0)$, then the perpendicular line is parallel to the y-axis – it has an undefined gradient.
- If a line is parallel to the y-axis then the perpendicular line is parallel to the x-axis – it has a gradient of zero.

EXAMPLES

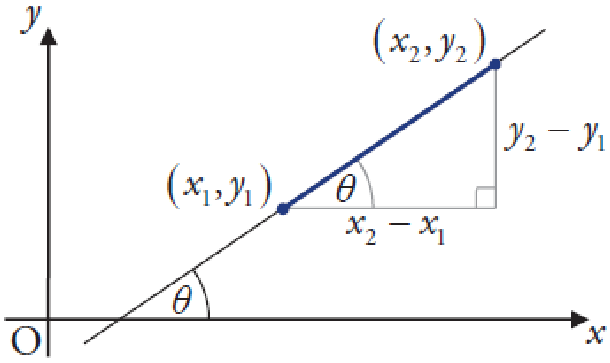
1. Given that T is the point $(1, -2)$ and S is $(-4, 5)$, find the gradient of a line perpendicular to ST.

2. Triangle MOP has vertices $\mathrm{M}(-3, 9)$, $\mathrm{O}(0, 0)$ and $\mathrm{P}(12, 4)$.
   Show that the triangle is right-angled.

Gradients and Angles (m=tan(θ))

Gradients A

Consider a straight line passing through the points (x_1, y_1) and (x_2, y_2):



Note
"θ" is the Greek letter "theta".
It is often used to stand for an angle.

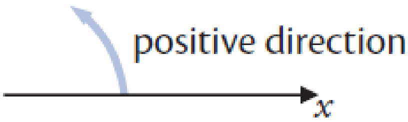
The **gradient** m of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{change in vertical height}}{\text{change in horizontal distance}} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{for } x_1 \neq x_2.$$

Also, since $\tan\theta = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{y_2 - y_1}{x_2 - x_1}$ we obtain:

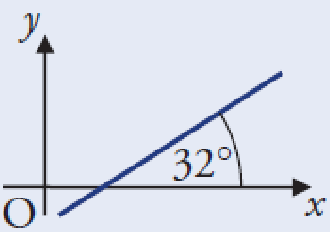
$$m = \tan\theta$$

where θ is the angle between the line and the positive direction of the x-axis.



EXAMPLES

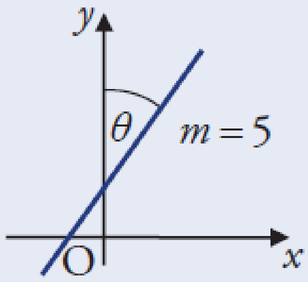
1. Calculate the gradient of the straight line shown in the diagram below.



Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

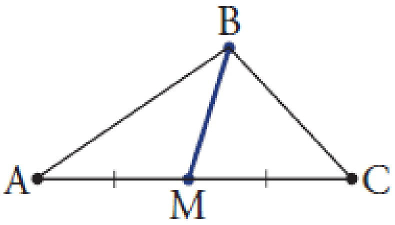
2. Find the angle that the line joining $P(-2,-2)$ and $Q(1,7)$ makes with the positive direction of the x-axis.

3. Find the size of angle θ shown in the diagram below.



Medians

A **median** of a triangle is a line through a vertex and the midpoint of the opposite side.



BM is a median of $\triangle ABC$.

The standard process for finding the equation of a median is shown below.

EXAMPLE

Triangle ABC has vertices $A(4, -9)$, $B(10, 2)$ and $C(4, -4)$.
Find the equation of the median from A.

Step 1
Calculate the midpoint of the relevant line.

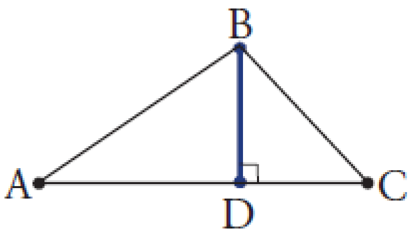
Step 2
Calculate the gradient of the line between the midpoint and the opposite vertex.

Step 3
Find the equation using this gradient and either of the two points used in Step 2.

1) Triangle ABC has vertices A(2, -7) , B(8, 4) and C(2, -2). Find the equation of the median from A.

Altitudes

An **altitude** of a triangle is a line through a vertex, perpendicular to the opposite side.



BD is an altitude of $\triangle ABC$.

The standard process for finding the equation of an altitude is shown below.

EXAMPLE

Triangle ABC has vertices $A(3, -5)$, $B(4, 3)$ and $C(-7, 2)$.

Find the equation of the altitude from A.

Step 1

Calculate the gradient of the side which is perpendicular to the altitude.

Step 2

Calculate the gradient of the altitude using $m \times m_{\perp} = -1$.

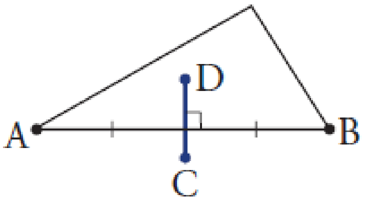
Step 3

Find the equation using this gradient and the point that the altitude passes through.

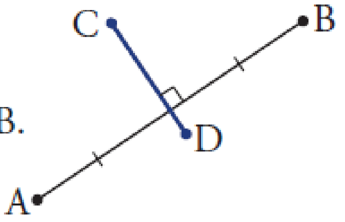
1) Triangle ABC has vertices A(1, -7) , B(2, 1) and C(-9, 0). Find the equation of the altitude from A.

Perpendicular Bisectors

A **perpendicular bisector** is a line which cuts through the midpoint of a line segment at right-angles.



In both cases, CD is the perpendicular bisector of AB.



The standard process for finding the equation of a perpendicular bisector is shown below.

EXAMPLE

A is the point $(-2, 1)$ and B is the point $(4, 7)$.

Find the equation of the perpendicular bisector of AB.

Step 1

Calculate the midpoint of the line segment being bisected.

Step 2

Calculate the gradient of the line used in Step 1, then find the gradient of its perpendicular bisector using $m \times m_{\perp} = -1$.

Step 3

Find the equation of the perpendicular bisector using the point from Step 1 and the gradient from Step 2.

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

1) Find the perpendicular bisector of the line joining A(3,-5) and B(5, -11).

Intersection of Lines

Many problems involve lines which intersect (cross each other). Once we have equations for the lines, the problem is to find values for x and y which satisfy both equations, i.e. solve simultaneous equations.

There are three different techniques and, depending on the form of the equations, one may be more efficient than the others.

We will demonstrate these techniques by finding the point of intersection of the lines with equations $3y = x + 15$ and $y = x - 3$.

Elimination

This should be a familiar method, and can be used in all cases.

$$3y = x + 15 \quad ①$$
$$y = x - 3 \quad ②$$

$$①-②: \quad 2y = 18$$
$$y = 9.$$

Put $y = 9$ into ②: $x = 9 + 3$
$= 12.$

So the lines intersect at the point $(12, 9)$.

Equating

This method can be used when both equations have a common x- or y-coefficient. In this case, both equations have an x-coefficient of 1.

Make x the subject of both equations:

$x = 3y - 15 \qquad x = y + 3.$

Equate:

$$3y - 15 = y + 3$$
$$2y = 18$$
$$y = 9.$$

Substitute $y = 9$ into:

$$y = x - 3$$
$$x = 9 + 3$$
$$= 12.$$

So the lines intersect at the point $(12, 9)$.

Substitution

This method can be used when one equation has an x- or y-coefficient of 1 (i.e. just an x or y with no multiplier).

Substitute $y = x - 3$ into:

$$\begin{aligned} 3y &= x + 15 \\ 3(x-3) &= x + 15 \\ 3x - 9 &= x + 15 \\ 2x &= 24 \\ x &= 12. \end{aligned}$$

Substitute $x = 12$ into:

$$\begin{aligned} y &= x - 3 \\ y &= 12 - 3 \\ &= 9. \end{aligned}$$

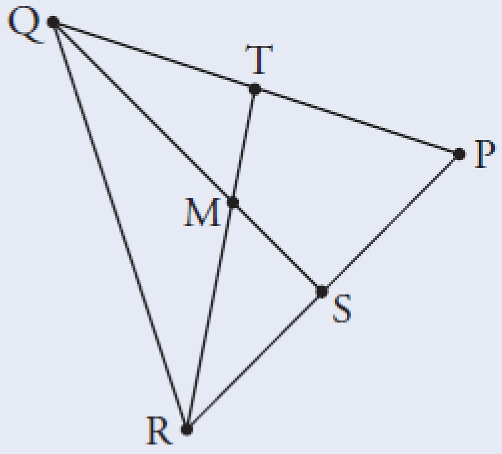
So the lines intersect at the point $(12, 9)$.

EXAMPLE

1. Find the point of intersection of the lines $2x - y + 11 = 0$ and $x + 2y - 7 = 0$.

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

2. Triangle PQR has vertices $P(8,3)$, $Q(-1,6)$ and $R(2,-3)$.



(a) Find the equation of altitude QS.

(b) Find the equation of median RT.

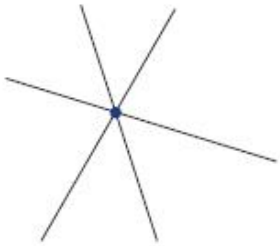
(c) Hence find the coordinates of M.

Concurrency

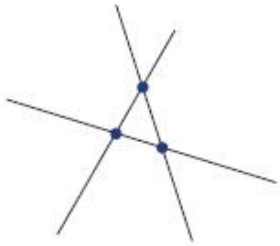
Any number of lines are said to be **concurrent** if there is a point through which they all pass.

So in the previous section, by finding a point of intersection of two lines, we showed that the two lines were concurrent.

For three lines to be concurrent, they must all pass through a single point.

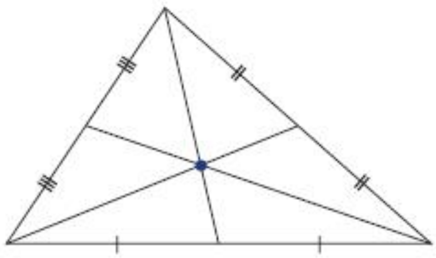


The three lines are concurrent

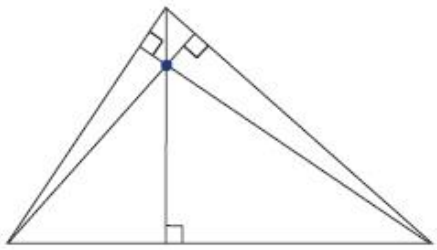


The three lines are not concurrent

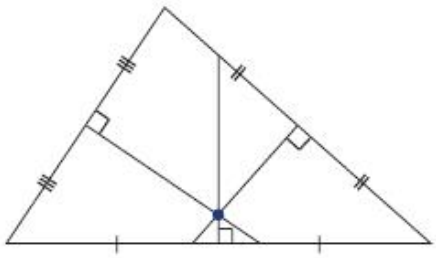
A surprising fact is that the following lines in a triangle are concurrent.



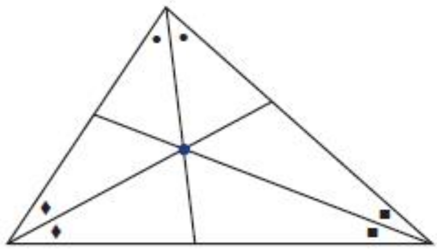
The three medians of a triangle are concurrent.



The three altitudes of a triangle are concurrent.



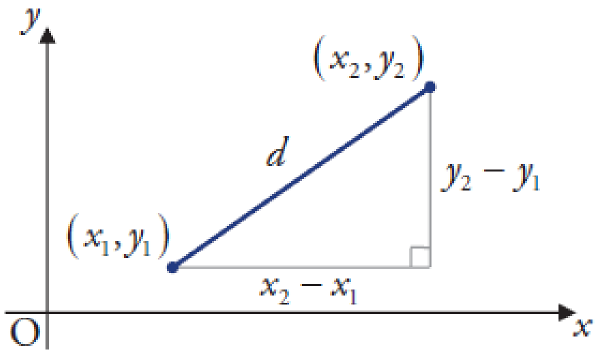
The three perpendicular bisectors in a triangle are concurrent.



The three angle bisectors of a triangle are concurrent.

The Distance Formula

The distance formula gives us a method for working out the length of the straight line between *any* two points. It is based on Pythagoras's Theorem.



Note
The "$y_2 - y_1$" and "$x_2 - x_1$" come from the method above.

The **distance** d between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ units.}$$

EXAMPLES

2. A is the point $(-2, 4)$ and $B(3, 1)$. Calculate the length of the line AB.

Note: We could just make a sketch and use Pythagoras' Theorem.

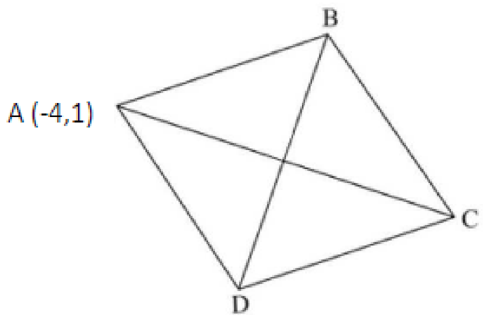
Practice Unit Assessments

Practice test 1

1 Find the equation of the line passing thorough (5, -10), parallel to the line with equation $4x + y - 8 = 0$. (2)

2. ABCD is a rhombus.
Diagonal BD has equation $y = 3x - 2$.
A has coordinates (–4, 1).

Find the equation of the diagonal AC

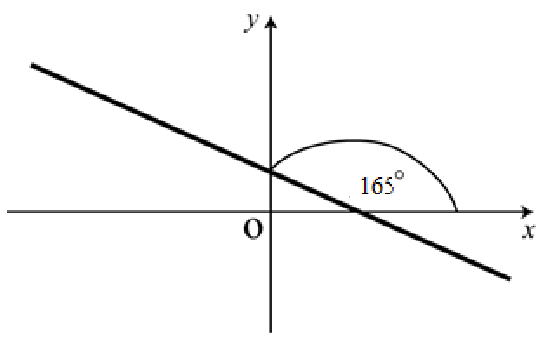


(#[2.1] and 1)

3 A ski slope is categorised by its gradient as shown in the table.

Dry slope category	Steepness (s) of slope
Teaching and general skiing	$0 \leq s \leq 0{\cdot}35$
Extreme skiing	$s > 0{\cdot}35$

(a) What is the gradient of the line shown in the diagram? **(1)**



(b) To which category does the ski slope represented by the line in part (a) belong?

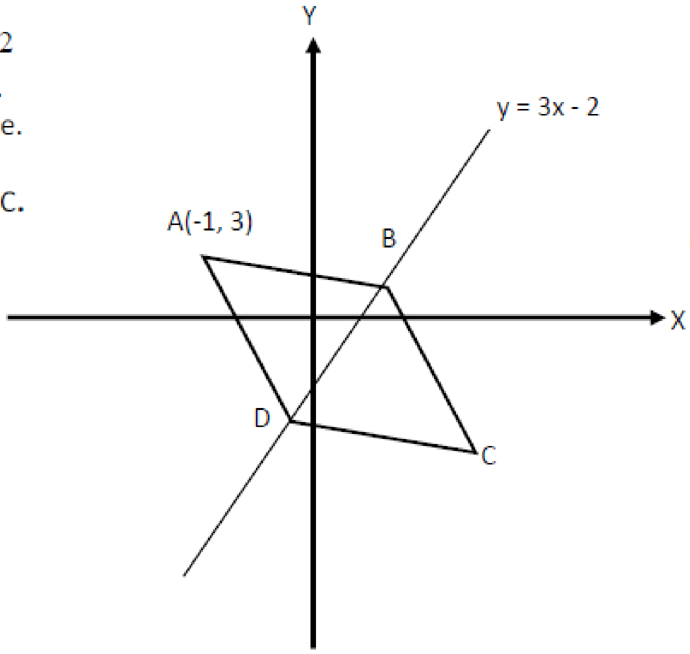
Give a reason for your answer. **(#[2.2])**

Practice test 2

1 A straight line has the equation $-6x + y - 2 = 0$.
Write down the equation of the line parallel to the given line, which passes through the point (3, –5).
(2)

2. ABCD is a rhombus.
Diagonal BD has equation $y = 3x - 2$
and point **A** has coordinates (–1, 3).
Note that the diagram is not to scale.

Find the equation of the diagonal AC.

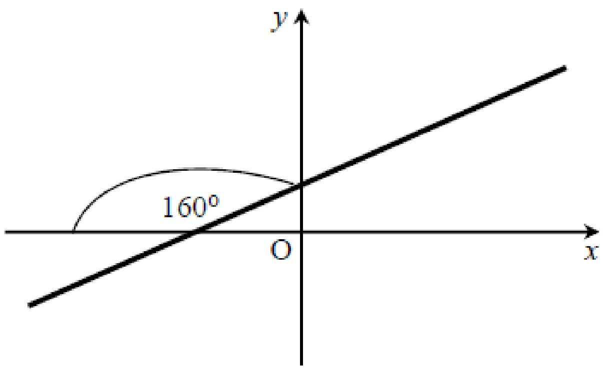


(#2.1 and 1)

3 A ramp is categorised by its gradient as shown in the table.

Category	Steepness (*s*) of ramp
Safe	$0 < s \leq 0.3$
Dangerous	$s > 0.3$

Which category does the ramp in the diagram below belong to?
Explain your answer fully.



(#2.2)

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Practice Unit Assessment Solutions

Applications 1.1

(1) $4x + y - 8 = 0$

$y = -4x + 8$

$m = -4$ • ✓

$y - b = m(x - a)$

$y + 10 = -4(x - 5)$ • ✓

$y + 10 = -4x + 20$

$y = -4x + 10$

(2) $m_{BD} = 3$

$m_{AC} = -\frac{1}{3}$ (as perpendicular) • ✓

$y - 1 = -\frac{1}{3}(x + 4)$ • ✓

$3(y - 1) = -(x + 4)$

$3y - 3 = -x - 4$

$3y = -x - 1$

(3)(a) $m = \tan\theta$

$m = \tan 165$

$m = -0.27$ • ✓

(b) in context of Q $s = 0.27$ It belongs Teaching & general skiing as $0 \le s \le 0.35$ • ✓

Test 2

(1) $-6x + y - 2 = 0$

$y = 6x + 2$

$m = 6$

$y - -5 = 6(x - 3)$

$y + 5 = 6x - 18$

$\underline{y = 6x - 23}$

(2) $m_{BD} = 3$

$m_{AC} = -\frac{1}{3}$ as $BD \perp AC$

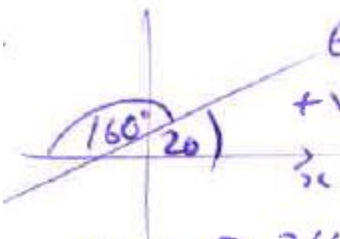
$y - -1 = -\frac{1}{3}(x - 3)$

$-3(y + 1) = x - 3$

$-3y - 3 = x - 3$

$\underline{-3y = x}$

(3) $m = \tan 20$



θ always from +ve direction of x-axis

$m = 0.364$

Dangerous as $0.364 > 0.3$